Ce1

Preamble to Complex Algebra

<u>Review of Functions of a real variable</u>

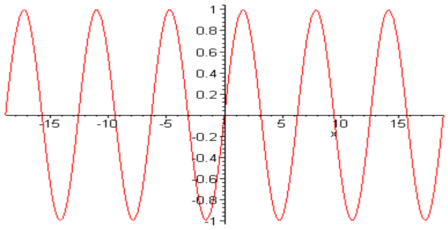
Circular (Trigonometric) functions

Polar: $x = r.\cos\theta$, $y = r.\sin\theta$

Circle: $x^2 + y^2 = r^2$

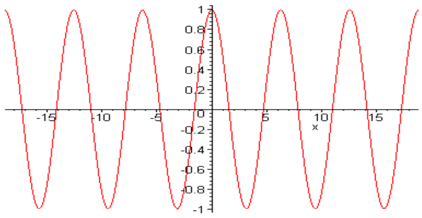
sinx: odd function $\sin(x) = -\sin(-x)$

Rotational symmetry (about origin)



cosx: even function $\cos(x) = \cos(-x)$

Reflectional symmetry (about y axis)



Properties: $\sin^2 x + \cos^2 x = 1$; $\sin 2x = 2.\sin x.\cos x$; $\cos 2x = \cos^2 x - \sin^2 x$;
$\sin(x + y) = \sin x.\cos y + \cos x.\sin y$; $\cos(x + y) = (\cos x.\cos y - \sin x.\sin y)$; etc. …
sin$x$ and cos$x$ are both <u>periodic</u> functions (period 2π).

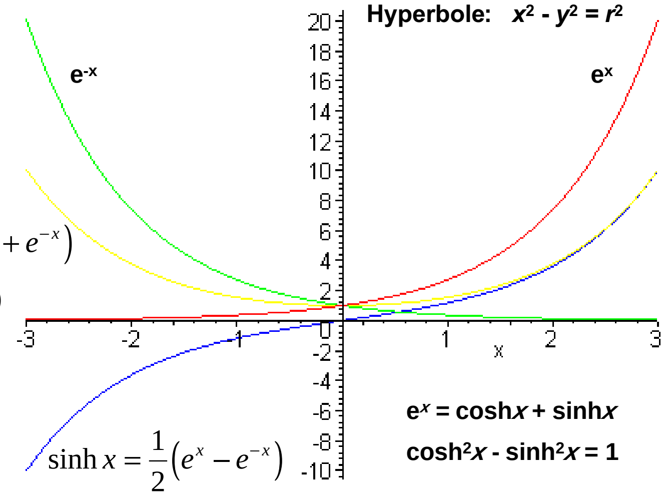
Ce2

Exponential & Hyperbolic Functions

Non-periodic functions

Parametric: $x = r.\cosh\theta$, $y = r.\sinh\theta$

Hyperbole: $x^2 - y^2 = r^2$



$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

(even function)

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

(odd function)

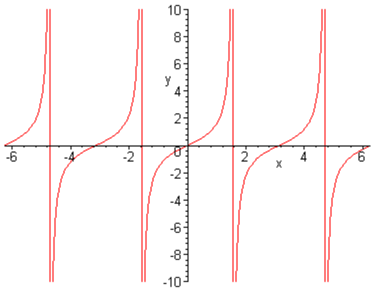
$e^x = \cosh x + \sinh x$

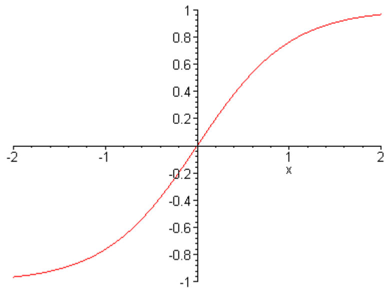
$\cosh^2 x - \sinh^2 x = 1$

Ce3 tan(x) and tanh(x)

$\tan x = \dfrac{\sin x}{\cos x}$ **(odd function)**

$\tanh x = \dfrac{\sinh x}{\cosh x}$ **(odd function)**





$$\tan 2x = \frac{2\tan x}{\left(1-\tan^2 x\right)}$$

$$\tanh 2x = \frac{2\tanh x}{\left(1+\tanh^2 x\right)}$$

tan*x* is a periodic function (period π)

tanh*x* is a non-periodic function

Ce4 Complex Algebra – History

Notation: $z = a + jb$ where $j^2 = -1$ (alternative: i used instead of j)

Cardano (1545) - dismissed complex numbers as "subtle as they are useless"

Bombelli (1572) - " . . . sophistry rather than truth"

Leibniz (1702) - "that amphibian between existence and nonexistence"

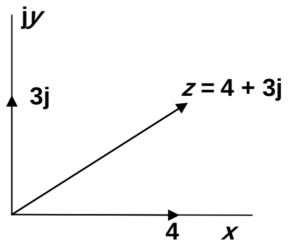
1770: the situation was still sufficiently confused for so great a mathematician as Euler to mistakenly argue: $\sqrt{-2}.\sqrt{-3} = \sqrt{6}$ which of course is wrong !!!

The situation was clarified at the end of the eighteenth century with:

Wessel, Argand, Gauss:

geometric interpretation ⟶ Complex Plane

Argand diagram



Complex analysis flourished 1814 - 1851

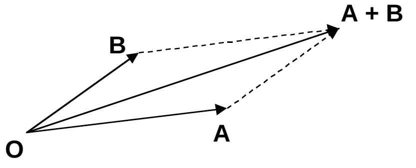
Cauchy, Riemann, . . .

Ce5 Geometric Operations

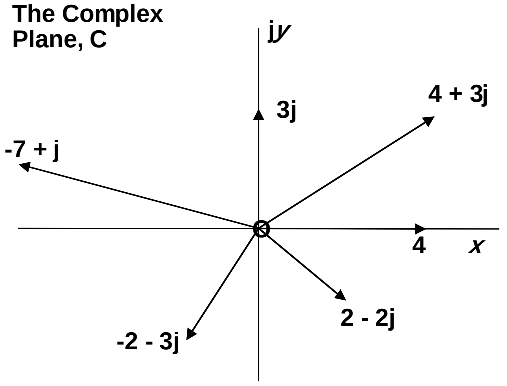
Addition and multiplication of complex numbers can now be given definite meanings as geometric operations:

Sum A + B

Parallelogram rule of ordinary vector addition:



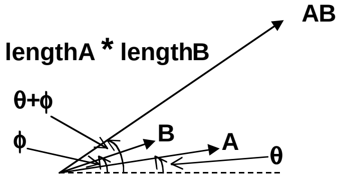
The Complex Plane, C



Product AB (much less obvious)

Length of AB is the product of the lengths of A and B

Angle of AB is the sum of the angles of A and B



e.g. 3.j = 3j

$j^2 = -1$

Ce6 Motivation for Complex Numbers

Motivation for complex numbers was NOT the solution of quadratic:

$$x^2 = mx + c$$

$$x = \frac{1}{2}\left[m \pm \sqrt{m^2 + 4c}\right]$$

$$\text{discriminant} \quad d = m^2 + 4c$$

Motivation WAS cubic: $x^3 = 3px + 2q$

Cardano showed:

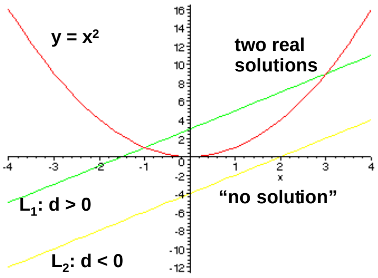
$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

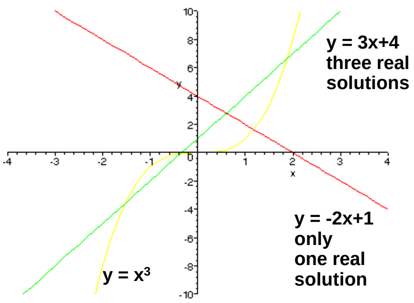
Bombelli considered: $x^3 = 15x + 4$

giving solution $x = 4$

But: (with $p = 5$ and $q = 2$)

$$x = \sqrt[3]{2 + j11} + \sqrt[3]{2 - j11}$$





Ce7 Algebraic Rules

Using plausible algebraic rules for addition and multiplication of complex numbers, we can show: $(2 \pm j)^3 = 2 \pm j11$

e.g. by $(2+j)^3 = (2+j)(2^2+4j+j^2) = (2+j)(3+4j) = (6+11j+4j^2) = 2+11j$

Hence, $x = \sqrt[3]{2+j11} + \sqrt[3]{2-j11} = (2+j)+(2-j) = 4$

Rules:

Addition: $(x_1+jy_1)+(x_2+jy_2) = (x_1+x_2)+j(y_1+y_2)$

Multiplication: $(x_1+jy_1)(x_2+jy_2) = (x_1x_2 - y_1y_2)+j(x_1y_2+x_2y_1)$

Polar notation: $x = r\cos\theta,\ y = r\sin\theta$

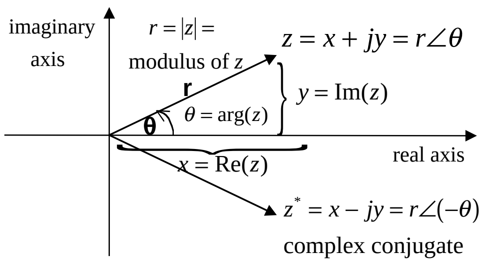
$z = x+jy = r\angle\theta = r(\cos\theta + j\sin\theta)$

$z_1+z_2 = (x_1+x_2)+j(y_1+y_2)$

$z_1z_2 = (r_1\angle\theta_1)(r_2\angle\theta_2) = r_1r_2\angle(\theta_1+\theta_2)$

$\ldots = r\angle(\theta-4\pi) = r\angle(\theta-2\pi) = r\angle\theta =$

$r\angle(\theta+2\pi) = r\angle(\theta+4\pi) = \ldots$



Ce8 Properties of Complex Numbers

$\mathrm{Re}(z) = \frac{1}{2}[z+z^*];\quad \mathrm{Im}(z) = \frac{1}{2j}[z-z^*];\quad |z| = \sqrt{x^2+y^2};\quad \tan[\arg(z)] = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)};\quad zz^* = |z|^2;$

Defining $\frac{1}{z}$ by $\left(\frac{1}{z}\right)z = 1$, it follows that $\frac{1}{z} = \frac{1}{r\angle\theta} = \frac{1}{r}\angle(-\theta);\quad r\angle\theta = r(\cos\theta + j\sin\theta);$

$\frac{R\angle\phi}{r\angle\theta} = \frac{R}{r}\angle(\phi-\theta);\quad \frac{1}{x+jy} = \frac{x-jy}{(x+jy)(x-jy)} = \frac{x-jy}{x^2+y^2} = \frac{x}{x^2+y^2} - j\frac{y}{x^2+y^2};$

$(z_1+z_2)^* = z_1^*+z_2^*;\quad (z_1z_2)^* = (z_1^*)(z_2^*);\quad \left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*};$ Triangle inequality $|z_1+z_2| \le |z_1|+|z_2|;$

Generalised triangle inequality $|z_1+z_2+\ldots+z_n| \le |z_1|+|z_2|+\ldots+|z_n|;$

When does equality hold?

Product computations: $(1+j)^4 = -4;\quad (1+j)^{13} = -2^6(1+j);\quad (1+j\sqrt{3})^6 = 2^6;$

Division computations: $\frac{(1+j\sqrt{3})^3}{(1-j)^2} = -4j;\quad \frac{(1+j)^5}{(\sqrt{3}+j)^2} = -\sqrt{2}\angle(-\pi/12);$

Ce9 Euler's Formula

Complex number: $z = r\angle\theta = r(\cos\theta + j\sin\theta)$

Euler's formula: $\cos\theta + j\sin\theta = e^{j\theta}$ miraculous !!!

Now we can write: $z = re^{j\theta}$

Also, the geometric rule for multiplying complex numbers now looks almost obvious: $(\mathrm{Re}^{j\phi})(re^{j\theta}) = Rre^{j(\phi+\theta)}$

An immediate consequence of the above is De Moivre's theorem:

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

since $(e^{j\theta})^n = e^{jn\theta}$

and more generally, $z^n = (re^{j\theta})^n = r^n e^{jn\theta} = r^n(\cos n\theta + j\sin n\theta)$

Ce10 Trig. derivation of De Moivre

Recall $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

and $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

In particular, $\cos 2\theta = \cos^2\theta - \sin^2\theta$; and $\sin 2\theta = 2\sin\theta\cos\theta$

Expanding $(\cos\theta + j\sin\theta)^2$

$$\begin{aligned}(\cos\theta + j\sin\theta)^2 &= \cos^2\theta + 2j\sin\theta\cos\theta + j^2\sin^2\theta \\ &= \cos^2\theta - \sin^2\theta + j2\sin\theta\cos\theta \\ &= \cos 2\theta + j\sin 2\theta\end{aligned}$$

The case for general power n is easily established using proof by induction:

viz. assuming $(\cos\theta + j\sin\theta)^m = \cos m\theta + j\sin m\theta$

$$\begin{aligned}(\cos\theta + j\sin\theta)^{m+1} &= (\cos m\theta + j\sin m\theta)(\cos\theta + j\sin\theta) \\ &= [\cos m\theta\cos\theta - \sin m\theta\sin\theta + j(\sin m\theta\cos\theta + \cos m\theta\sin\theta)] \\ &= \cos(m+1)\theta + j\sin(m+1)\theta\end{aligned}$$

Ce11 Power Series argument

Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

We suppose we can express e^x as a power series:

$$e^x = a_0 + a_1 x + a_2 x^2 + ... = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^r}{r!} + ...$$

[try differentiating to justify this expression for e^x].

$$\text{Hence, } e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + ... = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + ...$$

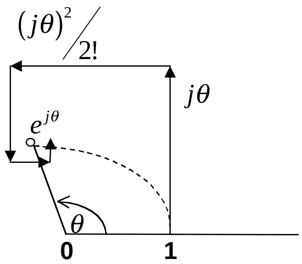
Spiralling series for $e^{j\theta}$ but it is not clear that it converges to a point on unit circle at angle θ.

Split spiral into real and imaginary parts:

$$e^{j\theta} = C(\theta) + jS(\theta)$$

where $C(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...;$ $S(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...$

even function odd function



Ce12 Sine and Cosine

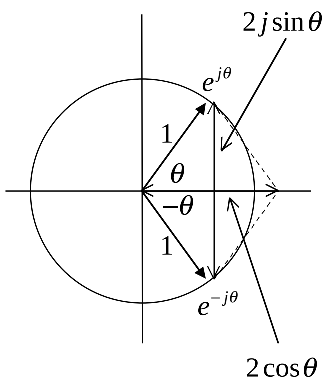
$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

Hence, $2\cos\theta = e^{j\theta} + e^{-j\theta}$

and $2j\sin\theta = e^{j\theta} - e^{-j\theta}$

$\therefore$ $\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$ and $\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$



Recall that for real valued x, hyperbolic functions

$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ and $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

These results give a hint of a remarkable relationship between the trigonometric and hyperbolic functions of complex numbers (see later).

Ce13 Fundamental Theorem of Algebra

The polynomial equation: $z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 = 0$
in variable z has exactly n complex number roots: $z_1, z_2, ..., z_n$
(although some of these roots may be repeated),
when the coefficients $a_{n-1}, a_{n-2}, ..., a_1, a_0$ are complex numbers.

When the coefficients $a_{n-1}, a_{n-2}, ..., a_1, a_0$ are real numbers,
then the polynomial equation: $z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 = 0$
in variable z also has n roots: $z_1, z_2, ..., z_n$ (again some may be repeated)
which may be real or complex numbers; and any such complex number roots always occur in complex conjugate pairs,
so that if $x + jy$ is a root, then so also is $x - jy$.
Note that if n is odd, then there is at least one real root.

Ce14 Application to Trigonometry

Useful Trigonometric relations arise directly from the Complex multiplication rule:

Let $C = \cos\theta,\quad S = \sin\theta,\quad c = \cos\phi,\quad s = \sin\phi$

We can view both $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$ as components of $e^{j(\theta+\phi)}$

$$\cos(\theta+\phi) + j\sin(\theta+\phi) = e^{j(\theta+\phi)} = e^{j\theta}e^{j\phi} = (C + jS)(c + js)$$
$$= (Cc - Ss) + j(Sc + Cs)$$

Hence, $\cos(\theta+\phi) = Cc - Ss$, also $\sin(\theta+\phi) = Sc + Cs$

$$\cos 3\theta + j\sin 3\theta = e^{j3\theta} = (e^{j\theta})^3 = (C + jS)^3 = C^3 + 3C^2 jS + 3Cj^2S^2 + j^3S^3$$
$$= C^3 + 3C^2 jS - 3CS^2 - jS^3 = (C^3 - 3CS^2) + j(3C^2S - S^3)$$

Using $C^2 + S^2 = 1$ we have

$$\cos 3\theta = C^3 - 3CS^2 = C^3 - 3C(1 - C^2) = 4C^3 - 3C = 4\cos^3\theta - 3\cos\theta$$
$$\sin 3\theta = 3C^2S - S^3 = 3(1 - S^2)S - S^3 = 3S - 4S^3 = 3\sin\theta - 4\sin^3\theta$$

Ce15

Applications

Also, since $2\cos\theta = e^{j\theta} + e^{-j\theta}$

$$2^4\cos^4\theta = (e^{j\theta} + e^{-j\theta})^4 = (e^{j4\theta} + e^{-j4\theta}) + 4(e^{j2\theta} + e^{-j2\theta}) + 6$$
$$= 2\cos 4\theta + 8\cos 2\theta + 6$$
$$\therefore \quad \cos^4\theta = \tfrac{1}{8}[\cos 4\theta + 4\cos 2\theta + 3]$$

Other Applications:

Geometry

Calculus e.g. finding the nth derivative of e^{ax}sinbx

Algebra e.g. evaluation of integrals such as: $\int \frac{dx}{x^n - 1}$

Vectorial operations: e.g. $a^* b = \underline{a} \bullet \underline{b} + j(\underline{a} \times \underline{b})$ in the Complex plane

Alternating currents in Electrical Networks

Stability in Control

Eigenvalues of matrices

Solution of differential equations

Ce16

Principal value of argument

The notation $z = r\angle\theta$ is now completely superceded by $z = re^{j\theta}$

So $z = re^{j\theta} = r(\cos\theta + j\sin\theta)$ with $|z| = r$ and $\arg(z) = \theta$

But $\arg(z) = \theta$ is only unique up to multiples of 2π, so more generally,

$z = r[\cos(\theta \pm 2k\pi) + j\sin(\theta \pm 2k\pi)] = re^{j(\theta \pm 2k\pi)}, \quad k = 0,1,2,...$

This more general representation is very important when considering roots.

When the angle lies in the interval $(-\pi, \pi]$ corresponding to $k = 0$

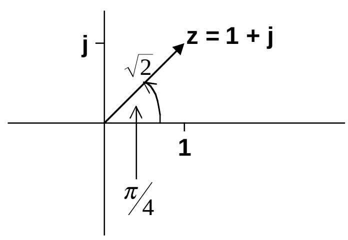
it is called the Principal value of argument z, $\theta = Arg(z), \ -\pi < Arg(z) \le \pi$

Principal value example:

$z = 1 + j = \sqrt{2}e^{j\pi/4} = \sqrt{2}[\cos(\pi/4) + j\sin(\pi/4)]$

$|z| = \sqrt{2}; \ \arg(z) = \frac{\pi}{4} \pm 2k\pi, \quad k = 0,1,2,...$

$\mathrm{Arg}(z) = \frac{\pi}{4}$ (the principal value)



Ce17

Roots of Complex Numbers

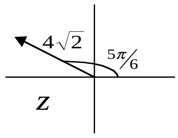
Let $z = re^{j\theta}$ and $w = \mathrm{Re}^{j\phi}$ Suppose that $z = w^n$

There are n distinct values (roots) of w, as follows:

$$w = \sqrt[n]{z} = \sqrt[n]{re^{j(\theta \pm 2k\pi)}} = \sqrt[n]{r}e^{j\frac{(\theta\pm 2k\pi)}{n}} = \sqrt[n]{r}[\cos\tfrac{(\theta\pm 2k\pi)}{n} + j\sin\tfrac{(\theta\pm 2k\pi)}{n}]$$

so that $R = \sqrt[n]{r}$ and $\phi = \dfrac{\theta}{n} \pm \dfrac{2k\pi}{n}$, for $k = 0,1,\dots,n-1$

Example:



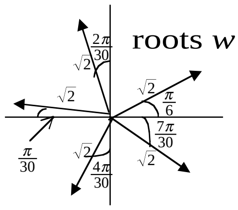
$$z = -2\sqrt{6} + j2\sqrt{2} = 4\sqrt{2}e^{j[\arctan(-\frac{1}{3}\sqrt{3})]} = 4\sqrt{2}e^{j(5\pi/6)}$$

$$w^5 = z = 4\sqrt{2}e^{j\left(\frac{5\pi}{6}\pm 2k\pi\right)},\ k = 0,1,2,\dots$$

$$w = \sqrt[5]{z} = \sqrt[5]{4\sqrt{2}e^{j\left(\frac{5\pi}{6}\pm 2k\pi\right)}} = \sqrt{2}e^{j\left(\frac{\pi}{6}\pm\frac{2k\pi}{5}\right)},\ k = 0,1,2,3,4$$

$$|w| = R = \sqrt{2},\ \phi = \arg(w) = \tfrac{\pi}{6}, \tfrac{17\pi}{30}, \tfrac{29\pi}{30}, \tfrac{41\pi}{30}, \tfrac{53\pi}{30}$$

Principal values, $Arg(w) = \tfrac{-19\pi}{30}, \tfrac{-7\pi}{30}, \tfrac{5\pi}{30}, \tfrac{17\pi}{30}, \tfrac{29\pi}{30}$



Ce18

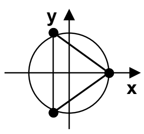
nth roots of unity

- ***n*th roots of *z*:** **The *n* values of** $w = \sqrt[n]{z}$ **lie on a circle of radius** $\sqrt[n]{r}$ **with centre at the origin and constitute the vertices of an *n*-sided regular polygon; the value with *k* = 0 is called the principal value.**

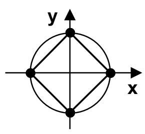
- ***n*th roots of unity:** $\sqrt[n]{1} = \cos\dfrac{2k\pi}{n} + j\sin\dfrac{2k\pi}{n},\quad k = 0,1,\dots,n-1$

Let ω denote the root of $\sqrt[n]{1}$ with $k = 1$

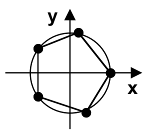
Then the n values are: $1,\ \omega,\ \omega^2,\ \dots,\ \omega^{n-1}$



$\sqrt[3]{1}$



$\sqrt[4]{1}$



$\sqrt[5]{1}$

Similarly, if w_1 is any n^{th} root of an arbitrary z, then the n values of $\sqrt[n]{z}$ are:

$w_1,\ w_1\omega,\ w_1\omega^2,\dots,w_1\omega^{n-1}$ since multiplying by ω^k corresponds to increasing the argument by $\dfrac{2k\pi}{n}$.

Ce19 Functions of complex numbers

$z = x + jy = re^{j\theta}$ Let $w = f(z) = \mathrm{R}e^{j\phi}$

$e^z = e^{x+jy} = e^x e^{jy}$ so that $R = e^x$ and $\phi = Arg(w) = y$

$\left|e^{jy}\right| = |\cos y + j\sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$ so that

$\left|\mathrm{e}^z\right| = e^x$ and $\arg(e^z) = y \pm 2k\pi, \quad k = 0,1,2,\ldots$

Trig functions, for real x $\cos x = \frac{1}{2}(e^{jx} + e^{-jx}),\ \sin x = \frac{1}{2j}(e^{jx} - e^{-jx})$

Define $\cos z = \frac{1}{2}(e^{jz} + e^{-jz}),\ \sin z = \frac{1}{2j}(e^{jz} - e^{-jz}),\ \tan z = \sin z / \cos z$ etc.

Solution of equations: $\cos z = 5,\ \cos z = \frac{1}{2}(e^{jz} + e^{-jz}) = 5 \ \therefore\ e^{jz} + e^{-jz} = 10$

$\left(e^{jz}\right)^2 - 10e^{jz} + 1 = 0$ (quadratic) $\therefore\ e^{jz} = \frac{10 \pm \sqrt{100-4}}{2} = 5 \pm \sqrt{24} = 9.899,\ 0.101$

Now $e^{jz} = e^{j(x+jy)} = e^{-y}e^{jx} = e^{-y}(\cos x + j\sin x);\ e^{-y} = 9.899,\ 0.101;\ e^{jx} = 1$

$y = \pm 2.292; \quad x = 2k\pi;$ so that $z = \pm 2k\pi \pm 2.292j, \quad k = 0,1,2,\ldots$

Ce20 Periodicity of exp z

- **Particular cases:**

$$e^{j2\pi} = 1;\quad e^{j\frac{\pi}{2}} = j;\quad e^{j\pi} = -1;\quad e^{-j\frac{\pi}{2}} = -j;\quad e^{-j\pi} = -1.$$

- **Modulus and argument:**

$$\left|e^{jy}\right| = |\cos y + j\sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

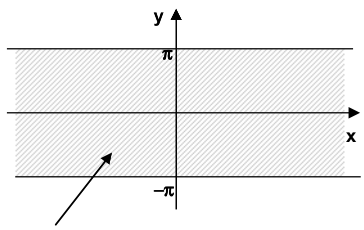
Hence, $\left|e^z\right| = e^x$ and $\arg\ e^z = y \pm 2n\pi, \quad n = 0,1,2,\ldots$

Note that: $e^z \neq 0$ for all z

Periodicity of e^z with period $2\pi j$

so that: $e^{z+2\pi j} = e^z$ for all z

Hence, for $w = e^z$, $-\pi < y \leq \pi$



Fundamental region of exp z

Ce21 Properties of Trigonometric Functions

- **Properties:**

$$\left.\begin{aligned}\cos z &= \cos x \cosh y - j \sin x \sinh y \\ \sin z &= \sin x \cosh y + j \cos x \sinh y\end{aligned}\right\} \text{ periodic with period } 2\pi$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

Also tanz, cotz are periodic with period π

- **Modulus:**

$$\text{Note}: \ |\sin z|,\ |\cos z| \to \infty \ \text{ as } \ y \to \infty, \ \text{ whereas } |\sin x|,\ |\cos x| \le 1$$

- **Addition rules:**

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$
$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$$
$$\text{Also,} \quad \cos^2 z + \sin^2 z = 1$$

Ce22 Hyperbolic Functions

- **Definitions:**

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \quad \sinh z = \tfrac{1}{2}(e^z - e^{-z})$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}.$$

- **Properties:**

$$\cosh jz = \cos z, \quad \sinh jz = j \sin z$$

and, since cosh is even i.e. $\cosh(z) = \cosh(-z)$

and sinh is odd i.e. $\sinh(z) = -\sinh(-z)$

we have,

$$\cos jz = \cosh z, \quad \sin jz = j \sinh z$$

Ce23

Logarithm

- **Definition:**

natural logarithm of *z = x + jy* denoted ln *z* (or log *z*) is defined as the inverse of the exponential function.

$$w = \ln z, \quad z \neq 0, \quad \text{defined by} \quad e^w = z$$

Set $\quad w = u + jv, \quad \text{and} \quad z = re^{j\theta}$

$\therefore \quad e^w = e^{u+jv} = re^{j\theta} = e^u . e^{jv}$

whence, $\; e^u = r, \quad v = \theta, \quad$ so that :

$$\ln z = \ln r + j\theta, \quad (r = |z| > 0, \quad \theta = \arg z)$$

ln *z* is infinitely many valued

- **Principal value:**

$$Lnz = \ln|z| + jArgz, \quad z \neq 0$$

where, $Argz$ is the principal value of the argument of z

then, $\quad \ln z = Lnz \pm 2n\pi j, \quad n = 0, \; 1, \; 2, \; . \; . \; .$

Ce24

General Powers

- **Properties of logarithm:**

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2; \quad \ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2.$$

$$e^{\ln z} = z; \quad \ln(e^z) = z \pm j2n\pi, \quad n = 0,1,2,...$$

- **General Powers:**

$z^c = e^{c \ln z}, \quad (c \; \text{complex}, \; z \neq 0), \quad$ multi – valued

Principal value of $\; z^c = e^{cLnz}$

$c = n = ...,-2,-1,+1,+2,... \quad z^n$ is single – valued

$c = \dfrac{1}{n}, \quad n = 2,3,... \quad z^c = \sqrt[n]{z}, \quad n$ distinct values

$c = \dfrac{p}{q}, \quad$ quotient of +ve integers, similar

otherwise, $\; z^c$ infintely – many valued